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Forward and Backward Running Waves in the Arteries: Analysis Using the Method of Characteristics

The one-dimensional equations of flow in the elastic arteries are hyperbolic and admit nonlinear, wavelike solutions for the mean velocity, U, and the pressure, P. Neglecting dissipation, the solutions can be written in terms of wavelets defined as differences of the Riemann invariants across characteristics. This analysis shows that the product, dUdP, is positive definite for forward running wavelets and negative definite for backward running wavelets allowing the determination of the net magnitude and direction of propagating wavelets from pressure and velocity measured at a point in the artery. With the linearizing assumption that intersecting wavelets are additive, the forward and backward running wavelets can be separately calculated. This analysis, applied to measurements made in the ascending aorta of man, shows that forward running wavelets dominate during both the acceleration and deceleration phases of blood flow in the aorta. The forward and backward running waves calculated using the linearized analysis are similar to the results of an impedance analysis of the data. Unlike the impedance analysis, however, this is a time domain analysis which can be applied to nonperiodic or transient flow.

Introduction

The shape of the pressure and flow pulses in the large arteries is the result of a complex, dynamic interaction between the mechanical properties of the left ventricle and the systemic arteries. The wave nature of flow in the arteries, described by Young [1], has enabled this cardiovascular interaction to be studied in terms of the propagation of forward running waves originating in the ventricle and backward running, reflected waves.

The normal periodicity of the cardiac cycle has led to the use of Fourier analysis in the now well-established calculation of arterial impedance [2]. This calculation requires a number of assumptions; that there is a linear relation between the pressure and flow rate at each frequency, so that the separate harmonic components can be superimposed and, implicitly, that the system is in steady-state oscillation. These assumptions and the observation that the flow is in phase with the pressure in forward running waves and 180 deg out of phase in backward running waves allow one to calculate the magnitude of the oscillatory portion of the separated forward and backward running waves which constitute the measured pressure and flow [3–5].

The method of characteristics is an alternative mode of analysis of one-dimensional waves which has been used in analogous problems in gas dynamics [6] and does not

necessarily assume either linearity or periodicity. In this paper we will develop this mode of analysis and explore the potential value of its application to the arterial system with particular reference to measurements obtained in the ascending aorta of man.

Theory and Results

The one-dimensional equations of flow in an impermeable, uniform elastic tube can be written [7–11]

$$S_t + (US)_z = -W$$
$$U_t + UU_z = -P_z/\rho + F, \qquad (1)$$

where S is the cross-sectional area, U and P are the spatially averaged velocity and pressure, ρ is the density of the fluid, W is the volume flow rate per unit length out of the tube, F represents the net effect of shear stresses at the wall of the tube, z is the distance along the tube, t is time and subscripts denote partial differentiation. In general, the dissipation function $F(P,U) < 0$ for $U > 0$. If the area of the tube depends only upon the instantaneous local pressure, $S = A(P; z,t)$, then these equations can be written in terms of P and U

$$P_t + UP_z + \rho c^2 U_z = -\rho c^2 (W + A_t + UA_z)/A \qquad (2)$$
$$U_t + 1/\rho P_z + UU_z = F,$$

where $c^2 = A/\rho(dA/dP)$ is the square of the wave speed which, in general, is a function of the pressure [12]. These first-order

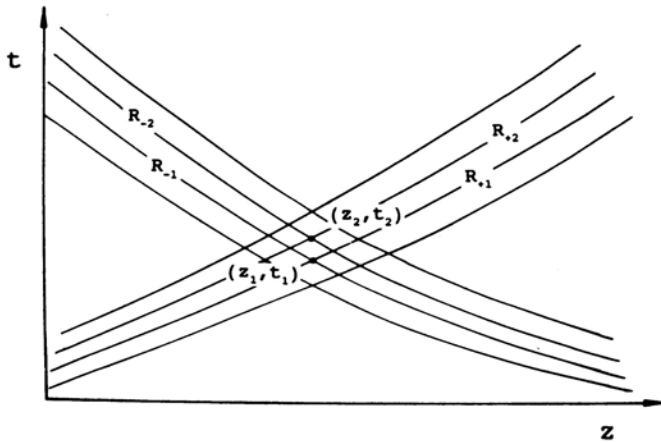**Fig. 1** A sketch of intersecting forward (R_+) and backward (R_-) running characteristics plotted in the $z-t$ plane. A unique pair of characteristics intersect at every point (z,t), forming an alternative coordinate system. As P and U measurements are usually taken at a fixed site, we show $z_2 = z_1$ although this is not a necessary condition. The difference operator, d, is defined as the difference between two characteristics.
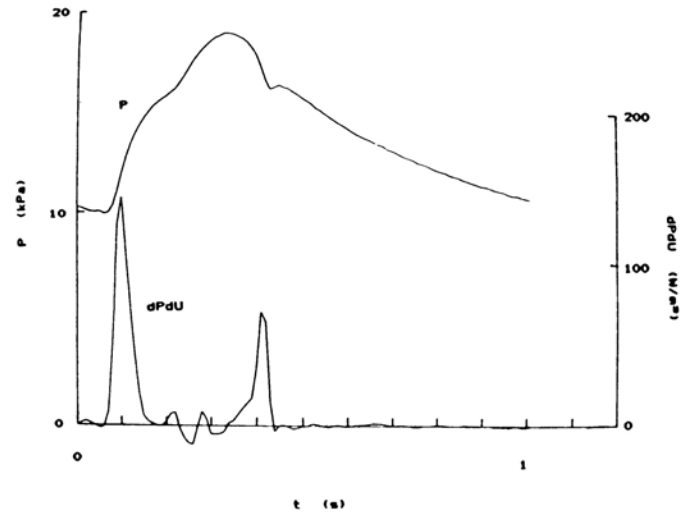

Fig. 2 The pressure, P, measured in the ascending aorta of a patient with normal cardiac function together with the rate of energy flux per unit area, $dPDU$, calculated from the measured data. The data shown are the ensemble average values calculated from 16 consecutive cardiac cycles using the Q-wave of the ECG to determine beat starts. The data were digitized with a sample period of 10 ms and the differences were calculated using a centered difference scheme.

partial differential equations are hyperbolic in nature and are therefore amenable to analysis by the method of characteristics. The characteristic directions for these equations are defined by, $dz/dt = U \pm c$, where the positive sign refers to the "forward" characteristic and the minus to the "backward" characteristic direction. Along these characteristic directions, the equations reduce to the ordinary differential equations

$$\frac{dU}{dt} \pm \frac{1}{\rho c} \frac{dP}{dt} = F \mp \frac{c}{A}(W + A_t + UA_z)$$

$$\text{on } \frac{dz}{dt} = U \pm c. \qquad (3)$$

If $c = c(P)$, these equations can be written in terms of the Riemann functions, $R_\pm$

$$R_\pm = U \pm \int_{P_o}^{P} \frac{dP}{\rho c}. \qquad (4)$$

Previously, these equations have been used primarily to predict the development of the propagating pulse wave which involves the evaluation of the right hand side of equation (3). In this work we take a slightly different point of view, using the equations to interpret measured changes of pressure and velocity at a fixed point in terms of the infinitesimal wavefronts, or "wavelets," coincident at the point of measurement. In regions away from branches and arterial discontinuities, it is reasonable to assume that viscous losses and flow out of the artery are negligible locally and that the artery is uniform and constant in its properties. In this case, which we assume in the rest of the paper, the right-hand side of equation (3) is zero and $R_\pm = $ constant along the characteristic directions are the most general solutions of these equations and $R_\pm$ are called the Riemann invariants.

Any changes in pressure or velocity which are imposed on the flow will cause a change in $R_\pm$ which will propagate along their characteristic directions. Consider two waves intersecting, looking in particular at the two pairs of characteristics which intersect at (z_1, t_1) and (z_2, t_2) (Fig. 1). In the forward running wave the two characteristics are associated with two Riemann invariants R_{+1} and R_{+2}, and from their definition we can write

$$R_{+2} - R_{+1} = U_2 - U_1 + \int_{P_1}^{P_2} \frac{dP}{\rho c} \qquad (5)$$

Similarly, the two backward running characteristics are associated with the two Riemann invariants, R_{-2} and R_{-1}, and from their definition

$$R_{-2} - R_{-1} = U_2 - U_1 - \int_{P_1}^{P_2} \frac{dP}{\rho c} \qquad (6)$$

If the waves are continuous we can define the difference operator, d, as $df = f(z_2, t_2) - f(z_1, t_1)$ and write

$$dR_\pm = dU \pm dP/\rho c. \qquad (7)$$

These two equations can be solved for dP and dU

$$dP = \rho c(dR_+ - dR_-)/2 \qquad (8)$$

$$dU = (dR_+ + dR_-)/2.$$

The product

$$dPdU = \rho c(dR_+^2 - dR_-^2)/4 \qquad (9)$$

has the potentially useful property that forward running wavelets, both compression ($dP > 0$) and expansion ($dP < 0$), make a positive contribution to the product while backward running wavelets make a negative contribution.

$dPdU$ has the dimensions of rate of energy flux per unit area, W/m^2, and corresponds to acoustic intensity [10]. It refers to the energy flux associated with the wave motion only and is a much smaller quantity than the flow power per unit area, PU.

Figure 2 shows the results of this analysis applied to the blood pressure and velocity measured by catheter mounted pressure and electromagnetic velocity sensors positioned in the ascending aorta of a normal man [13]. At this fixed site, dP and dU were calculated as the differences between P and U measured at 10 ms intervals. The figure includes the ensemble average pressure and velocity for 16 consecutive beats taking the Q-wave of the ECG as the starting point for each beat. There are two distinct positive peaks of $dPdU$ during left ventricular ejection. The first peak occurs during early systole and represents a forward running compression wave produced by the initial contraction of the left ventricle. After a period of about 250 ms, there is a second positive peak which corresponds to a forward running expansion wave. This wave, also of left ventricular origin, causes deceleration resulting in flow reversal and aortic valve closure. Apart from these two
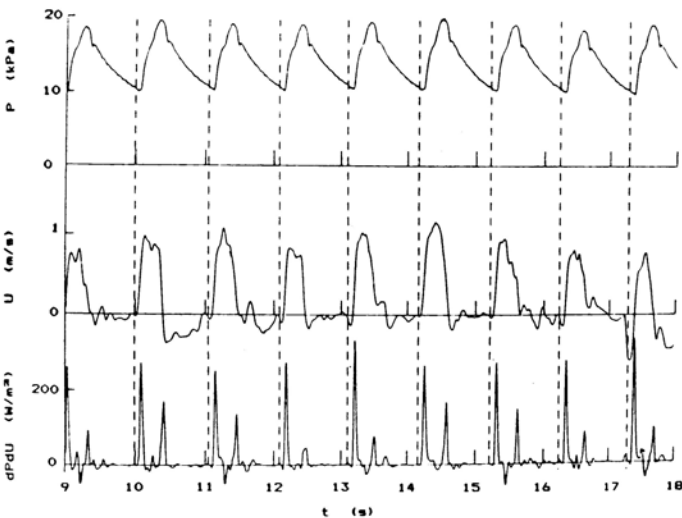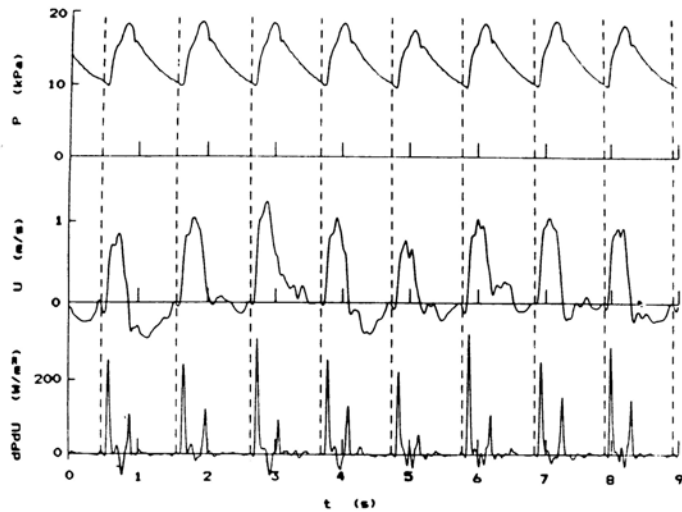
Fig. 4 The ratio dU/dP calculated from the data shown in Fig. 2 during the two peaks in dPdU. When forward running waves dominate, this ratio is equal to 1/ρc. The line is fitted by eye and corresponds to a wave speed of 4.8 m/s.

Fig. 3 The measured pressure, P, and velocity, U, together with the calculated rate of energy flux per unit area, dPdU. The dotted lines denote the Q-wave of the simultaneously measured ECG. These data are the basis of the ensemble averaged data shown in Fig. 2. It is noted that the velocity data are much "noisier" than that obtained using cuff type flow meters which may be the result of the lower signal to noise ratio of catheter mounted probes or movement of the catheter within the artery or may be the result of velocity disturbances present in the ascending aorta.

dominating forward running waves, relatively little net wave motion is indicated.

Figure 3 shows the result of this analysis applied to a portion of the continuous data from which the ensemble averages in the previous figure were obtained. The second positive peak of *dPdU*, corresponding to a forward running expansion wave, is consistently present but varies in magnitude from beat to beat. A smaller mid-systolic negative peak, corresponding to a backward running, or reflected wave, is apparent in a number of beats. The lack of a more distinct negative peak in the ensemble averaged data in Fig. 2 probably indicates the variability of the time of arrival of the reflected wavelets.

Despite its simplicity, this mode of analysis is quite general. The nonlinear nature of the flow has been retained and the wave speed is dependent upon the instantaneous pressure. Also, the analysis of wavelets is not particularly restricting since any finite waveform may be considered as the sum of successive wavelets, although the mean pressure cannot be assigned to either the forward or the backward wave.

When a wavelet encounters a discontinuity in the tube properties, a similar analysis across the discontinuity shows that both a reflected and a transmitted wavelet may result [10]. The nature of the reflected wavelet and the relative magnitude
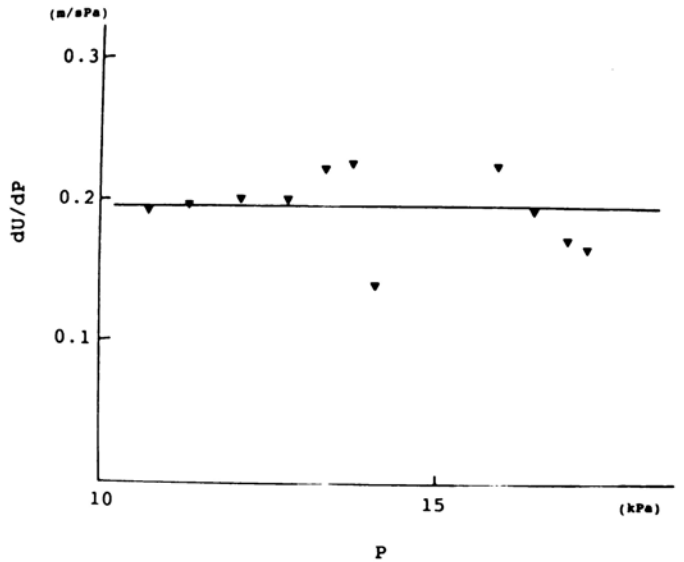
of the reflected to the transmitted wavelet will depend upon the type and severity of the discontinuity. The wave pattern formed by wavelet reflection and re-reflection in the arterial system may be very complex and may explain the difficulties encountered in following the propagation of individual wavelets.

However, considering an isolated wavelet propagating into a region of uniform conditions, the relationship between the pressure and the velocity changes across the wavelet follows from the conservation of either mass or momentum

$$dP_\pm = \pm \rho c \, dU_\pm. \tag{10}$$

It is thus possible to obtain an estimate of the pressure dependent speed of propagation of a wavelet in the artery during periods when wavelet travel is essentially unidirectional. Figure 4 shows the ratio *dU/dP* as a function of the instantaneous pressure calculated during the positive peaks of *dPdU* in Fig. 2 when forward running wavelets predominate. There are relatively few points as the periods of unidirectional wavelet travel were brief. The points at lower pressure correspond to the initial compression wave while those at higher pressure correspond to the expansion wave. In this patient the wave speed changed little between peak and trough pressures and so a constant wave speed was assumed. The line in the figure was fitted by eye and, assuming that the density of blood is 1.04×10^3 kg/m³, corresponds to a wave speed of 4.8 m/s.

Up to this point, the analysis has retained the nonlinear nature of the basic equations. With the further assumption that the changes in pressure and velocity associated with the waves are additive when they intersect, that is $dP = dP_+ + dP_-$ and $dU = dU_+ + dU_-$, it is possible to separate the measured P and U into forward and backward running components. This assumption is essentially the linearizing acoustic assumption of gas dynamics and will not be generally true for finite waves. Substituting into equation (8), the changes in pressure and velocity associated with the forward and backward running wavelets can be calculated from the measured data

$$dP_\pm = (dP \pm \rho c \, dU)/2 \tag{11}$$

$$dU_\pm = (dU \pm dP/\rho c)/2.$$

If ρc is taken to be the characteristic impedance, the integrated form of these equations are identical to those derived previously using impedance arguments [14].
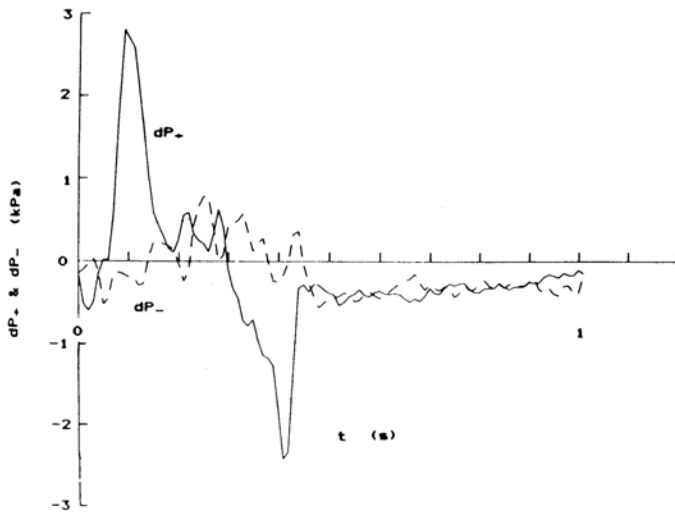
Fig. 5 The incremental pressure changes in the (———)-forward and (-----)-backward running wavelets calculated from the data shown in Fig. 2 using equation (11)

Fig. 6 A comparison of the composite forward and backward running waves calculated by the method of characteristics and by an impedance analysis. $P - ($———$)$ is the ensemble averaged pressure. $P_+ - ($———$)$ and $P_- - ($----$)$ are the integrated forward and backward running waves calculated from the data shown in Fig. 5. $P_f - ($———$)$ and $P_b - ($----$)$ are the forward and backward running waves calculated for the same data using the method of Westerhof et al. [4]. The scale refers to the measured pressure. The scale of the other curves is the same but because the absolute value is arbitrary in both analyses, the absolute value of the ordinate is suppressed.

Figure 5 shows the incremental pressure changes for the forward and backward running wavelets calculated from equation (11) using the value of $1/\rho c$ determined from Fig. 4 as described above. The finite forward and backward running pressure waves obtained by integrating the incremental curves are shown in Fig. 6. Also shown in Fig. 6 are the forward and backward running waves calculated from the same data using an impedance analysis [4]. The characteristic impedance, Z_o, was taken as the minimum impedance rather than the average of the higher frequency impedances as suggested by Westerhof et al., which, due to the relatively noisy velocity data, gave unreasonable results. The results of our impedance calculations are similar to those of other studies [15, 16] and resemble the results we have obtained using the method of characteristics.

Discussion

The analysis presented above, although simple, is general and may have advantages over other modes of analysis. The one-dimensional equations of motion for the fluid, equation (1), are based on several assumptions of which possibly the most limiting is the assumption of uniform velocity across the tube. If the velocity profile is not flat, extra terms depending upon the shape of the velocity profile appear on the right hand side of the momentum equations [17]. The one-dimensional equations retain the nonlinear nature of the flow and their solution in terms of the Riemann functions, equation (4), is general. The Riemann functions are constant along the characteristic directions only for the special case of a uniform, impermeable tube with negligible dissipation in which case they are usually called the Riemann invariants. These assumptions would be overly restrictive if applied to the arterial system as a whole but analysis of pressure and velocity measured at a single point requires only local validity. That is, as long as the assumptions are valid in the neighbourhood of the measurement site the analysis is valid.

The application of the theory to wavelets is also general since any finite wave can be described as the resultant of wavelets. Equation (7) should not be taken as a linearization of the analysis since $c = c(P)$. If the wavelet analysis is used to follow finite waves, this dependence of the wavelet speed on the local pressure may lead to changes in the waveform as the wave propagates and can explain the steepening and amplification of the pressure pulse in the systemic arteries [7].

We regard the calculation of the rate of energy flux per unit area from pressure and velocity measured at a fixed point by equation (9) as an important result of this analysis. The
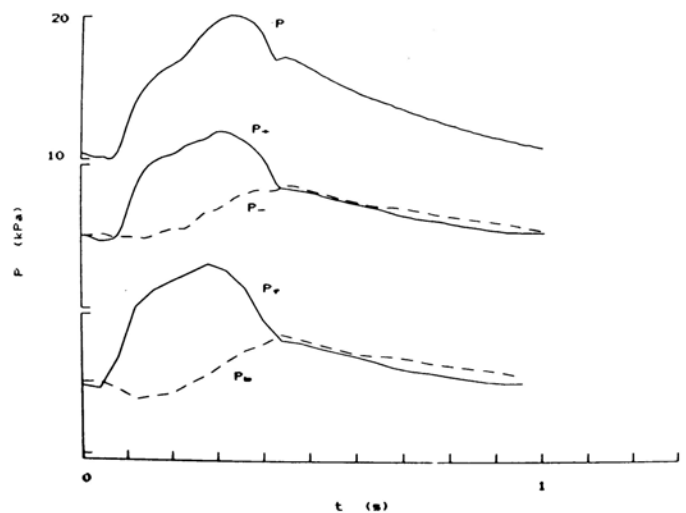
positive contribution of forward running waves and the negative contribution of backward running waves provides a simple means of evaluating the relative importance of forward and backward wavelets from measurements taken at a single point. The quantity $dPdU$ can be calculated directly from the measured pressure and velocity. It requires no assumption of linearity, periodicity or knowledge of the wave speed and, using suitable differentiating filters, may be performed in real time.

The separation of the measured pressure and velocity into forward and backward components using the method of characteristics does, however, require knowledge of the wave speed. During periods when either forward or backward running wavelets predominate, the wave speed can be determined from equation (10). In general, $c = c(P)$, and the calculation of dU/dP during the initial compression wave could provide a convenient measure of local wave speed as a function of pressure. The results for the ensemble averaged data shown in Fig. 4 suggest that the wave speed may be constant over the range of pressures measured in this particular subject.

The further more restricting assumption that coincident forward and backward running wavelets are simply additive enables the separate calculation of the forward and backward running wavelets from the pressure and velocity measured at a single point. These assumptions linearize the problem and, as might be expected, the results shown in Fig. 6 are similar to those of the impedance analysis which also assumes linearity. The assignment of the mean pressure between the forward and backward running waves is arbitrary in both analyses and we have followed the convention of equal apportionment.

Analysis of arterial flow in terms of wavelets has a number of advantages. The method of characteristics which gives rise to the concept of a wavelet retains the nonlinear nature of the flow. The analysis is done in the time domain and can be applied to transient and nonperiodic phenomena. The prevailing direction of wavelet motion at any time is indicated by the sign of $dPdU$. Since the calculation of $dPdU$ involves only differences, it does not include work or energy contributions from the mean pressure and flow and so cannot be interpreted as a measure of the total work done by the heart. It is analogous to acoustic intensity and may be taken as a measure of that part of the fluid energy which is due to the existence of

wavelets. Whether or not $dPdU$ will be as useful a concept in arterial dynamics as acoustic intensity is in acoustics is an open question which this paper is intended to introduce, not answer.

The ratio dU/dP is a measure of the instantaneous wavelet speed during periods when wavelet motion is unidirectional. If validated by independent measurement, it would provide a convenient means of determining the local arterial wave speed from measurements taken at a single site. Thus, if the intersecting wavelets are additive, the forward and backward waves of pressure and velocity can be calculated, again in the time domain from measurements taken at a single site.

An initial application of this analysis to measurements of P and U in the ascending aorta of man indicates clearly that both the early systolic acceleration and the late systolic deceleration of the blood are caused by forward running wavelets originating in the left ventricle. The early systolic compression wave may be the initial ventricular impulse [18] but the dominance of the forward running expansion wave in late systole has not, to our knowledge, been discussed. We speculate that it would be advantageous if the energy of deceleration was recovered, stored and subsequently released by elastic recoil of the ventricular wall, so facilitating left ventricular filling [19].

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